The Three-Dimensional Gaussian Product Inequality

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based on joint work with Ze-Chun Hu and Guolie Lan

Outline

1. Introduction
2. Improved Gaussian product inequalities for special cases
   - The symmetric case: $H_{n,n}(\gamma) \geq 0$
   - The asymmetric case: $H_{m,n}(\gamma) > 0$
3. Proof of 3-D Gaussian product inequality and extension

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Inequalities involving Gaussian distributions are related to various fields and have attracted great concern.

Royen (14): Gaussian correlation inequality.

For any closed symmetric sets K, L in $\mathbb{R}^d$ and any centered Gaussian measure μ we have

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

Gaussian product conjecture

For any d-dimensional real-valued centered Gaussian random vector $(X_1, \ldots, X_d)$,

$$\mathbf{E}[X_1^{2m} X_2^{2m} \cdots X_d^{2m}] \geq \mathbf{E}[X_1^{2m}]\mathbf{E}[X_2^{2m}] \cdots \mathbf{E}[X_d^{2m}], \quad m \in \mathbb{N}.$$

Real polarization problem

For any $d \geq 2$, and any collection $x_1, \ldots, x_d$ of unit vectors in $\mathbb{R}^d$, there exists a unit vector $v \in \mathbb{R}^d$ such that

$$|\langle v, x_1 \rangle \cdots \langle v, x_d \rangle| \geq d^{-d/2}.$$

As a consequence, for $d \geq 2$ and for every real Hilbert space $\mathcal{H}$ of dimensional at least d, one has that

$$\inf\{M > 0 : \forall u_1, \ldots, u_d \in S(\mathcal{H}), \exists v \in S(\mathcal{H}) : |\langle u_1, v \rangle \cdots \langle u_d, v \rangle| \geq M^{-1}\} = d^{d/2},$$

and $S(\mathcal{H}) := \{u \in \mathcal{H} : \|u\|_{\mathcal{H}} = 1\}$.

U-conjecture

Let $X = (X_1, \ldots, X_d)$ be a Gaussian vector such that $X \sim \mathcal{N}(0, I_d)$. If two polynomials $P(X)$ and $Q(X)$ are independent, then they are unlinked.

$P(X)$ and $Q(X)$ are said to be unlinked if there exist an isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ and an index $r \in \{1, \ldots, d-1\}$ such that $P(X) \in \mathbb{R}[Y_1, \ldots, Y_r]$ and $Q(X) \in \mathbb{R}[Y_{r+1}, \ldots, Y_n]$, where $Y = (Y_1, \ldots, Y_d) = T(X)$.

Li and Wei (12): Improved version of the Gaussian product conjecture:

$$\mathbf{E}\left[\prod_{j=1}^{d}|X_j|^{\alpha_j}\right] \geq \prod_{j=1}^{d}\mathbf{E}[|X_j|^{\alpha_j}],$$

where $\alpha_j, j = 1, 2, \ldots, d$, are nonnegative real numbers.

No universal method is available for proving the Gaussian product conjecture.

Frenkel (08) used algebraic methods to give proof for the case $\alpha_j = 2$ $(m = 1)$.

Wei (14) used integral representations to prove that for $\alpha_j \in (-1, 0)$,

$$\mathbf{E}\left[\prod_{j=1}^{d} |X_j|^{\alpha_j}\right] \geq \mathbf{E}\left[\prod_{j=1}^{k} |X_j|^{\alpha_j}\right] \mathbf{E}\left[\prod_{j=k+1}^{d} |X_j|^{\alpha_j}\right].$$

Stronger version of Gaussian product inequality does not necessarily hold in general.

Let U and V be independent standard Gaussian random variables. Since

$$\mathbf{E}\left[U^2(U+2V)^2(U-2V)^2\right] = \mathbf{E}\left[U^6 - 8U^4V^2 + 16U^2V^4\right] = 39,$$

and

$$\mathbf{E}[U^2]\mathbf{E}\left[(U+2V)^2(U-2V)^2\right] = \mathbf{E}\left[U^4 - 8U^2V^2 + 16V^4\right] = 43,$$

we have

$$\mathbf{E}\left[U^2(U+2V)^2(U-2V)^2\right] < \mathbf{E}[U^2]\mathbf{E}\left[(U+2V)^2(U-2V)^2\right].$$

Ornstein-Uhlenbeck operator on $\mathbb{R}^d$:

$$\mathcal{L}f = \Delta f - \langle x, \nabla f \rangle.$$

$\gamma_d = (2\pi)^{-d/2} \exp\{-|x|^2/2\} dx.$

$\text{Spectrum}(-\mathcal{L}) = \mathbb{N}$.

$\{H_k : k = 0, 1, \dots\}$: Hermite polynomials on $\mathbb{R}$.

$\text{Ker}(\mathcal{L} + kI)$: k-th eigenspace of $\mathcal{L}$:

$$F(x_1, \dots, x_d) = \sum_{i_1 + \cdots + i_d = k} \alpha(i_1, \dots, i_d) \prod_{j=1}^{d} H_{i_j}(x_j).$$

Malicet, Nourdin, Peccati and Poly (16) Fix $n \geq 1$, let $k_1, \ldots, k_n \geq 1$, and consider polynomials $F_i \in \mathrm{Ker}(\mathcal{L} + k_i I)$, $i = 1, \ldots, n$. Then,

$$\int_{\mathbb{R}^d} \left(\prod_{i=1}^{n} F_i^2 \right) d\gamma_d \geq \prod_{i=1}^{n} \int_{\mathbb{R}^d} F_i^2 d\gamma_d.$$

The equality holds if and only if the F_i's are jointly independent.

Karlin and Rinott (81) Gaussian product inequality holds for $\mathbf{X} = (X_1, \ldots, X_d)$ if the density of $|\mathbf{X}| = (|X_1|, \ldots, |X_d|)$ satisfies the condition of multivariate totally positive of order 2 ($\mathbf{MTP_2}$).

For any non-degenerate 2-dimensional centered Gaussian random vector (X_1, X_2), $(|X_1|, |X_2|)$ has a $\mathbf{MTP}_2$ density.

For a high dimensional ($d \geq 3$) centered Gaussian random vector $\mathbf{X}$, the density of $|\mathbf{X}|$ is not always $\mathbf{MTP}_2$.

Hu, Lan and Sun (19) For any 3-dimensional centered Gaussian random vector (X, Y, Z),

$$\mathbf{E}\left[X^{2m} Y^{2m} Z^{2m}\right] \geq \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2m}]\mathbf{E}[Z^{2m}], \quad \forall m \in \mathbb{N}.$$

The equality holds if and only if X, Y, Z are independent.

Intrinsic connection between moments of Gaussian distributions and the Gaussian hypergeometric functions.

New combinatorial identities and inequalities and more accurate lower bounds for some special cases.

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For $\alpha \in \mathbb{R}$, define

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1), & n \geq 1, \\ 1, & n = 0,\ \alpha \neq 0. \end{cases}$$

$n! = (1)_n$.

$$(2n-1)!! = 2^n \cdot \left(\frac{1}{2}\right)_n, \quad n \geq 0.$$

For $0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{(1)_n}{(1)_{n-k}(1)_k} = \frac{(1+n-k)_k}{(1)_k}.$$

Define

$$\binom{n}{k}_{\frac{1}{2}} := \frac{\left(\frac{1}{2}+n-k\right)_k}{\left(\frac{1}{2}\right)_k} = \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_{n-k}\left(\frac{1}{2}\right)_k} = \frac{(2n-1)!!}{(2n-2k-1)!!(2k-1)!!}.$$

$\binom{n}{k}_{\frac{1}{2}}$ may not be an integer. E.G. $\binom{4}{2}_{\frac{1}{2}} = \frac{35}{3}$ and $\binom{6}{3}_{\frac{1}{2}} = \frac{231}{5}$.

$$\binom{k+r}{r}_{\frac{1}{2}} \geq \binom{2}{1}_{\frac{1}{2}} = 3, \quad \forall k, r \in \mathbb{N}.$$

Theorem Let X and Y be independent centered Gaussian random variables. Then for any $r \in \mathbb{N}$ and $n, m \in \mathbb{N} \cup \{0\}$,

$$\mathbf{E}\left[X^{2m}Y^{2n}(X^2 - Y^2)^{2r}\right] \geq \binom{(m \wedge n) + r}{r}_{\frac{1}{2}} \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2n}]\left[\mathbf{E}(X+Y)^{2r}\right]^2.$$

The equality holds if and only if $m = n$ and $\mathbf{E}[X^2] = \mathbf{E}[Y^2]$.

$(X^2 - Y^2)^{2r} = (X+Y)^{2r}(X-Y)^{2r}$ and $\mathbf{E}[(X+Y)^{2r}] = \mathbf{E}[(X-Y)^{2r}]$.

Let $a^2 = \mathbf{E}[X^2]$ and $b^2 = \mathbf{E}[Y^2]$. Define

$$U = \frac{X}{a}, \quad V = \frac{Y}{b}.$$

Then U, V are independent standard Gaussian r.v.s.

Suppose that $m \geq n$. Then

$$\binom{(m \wedge n) + r}{r}_{\frac{1}{2}} = \frac{(2n + 2r - 1)!!}{(2n - 1)!!(2r - 1)!!}, \quad \mathbf{E}[(X+Y)^{2r}] = (2r-1)!!(a^2+b^2)^r.$$

$$\mathbf{E}\left[U^{2m}V^{2n}(a^2U^2 - b^2V^2)^{2r}\right] \geq (2m-1)!!(2n+2r-1)!!(2r-1)!!(a^2+b^2)^{2r}.$$

$$\mathbf{E}\left[U^{2m}V^{2n}\left(\gamma U^2-(1-\gamma)V^2\right)^{2r}\right]\geq 2^{m+n+2r}\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_{n+r}\left(\frac{1}{2}\right)_r,\ 0<\gamma<1.$$

For $\gamma\in\mathbb{R}$, define

$$G_{m,n}(\gamma)=\mathbf{E}\left[U^{2m}V^{2n}\left(\gamma U^2-(1-\gamma)V^2\right)^{2r}\right],$$

and

$$H_{m,n}(\gamma)=G_{m,n}(\gamma)-2^{m+n+2r}\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_{n+r}\left(\frac{1}{2}\right)_r.$$

To prove the improved Gaussian product inequality, it is sufficient to verify

$$H_{n,n}\left(\frac{1}{2}\right) = 0; \quad H_{n,n}(\gamma) > 0, \quad \gamma \in \left(0, \frac{1}{2}\right) \bigcup \left(\frac{1}{2}, 1\right);$$

$$H_{m,n}(\gamma) > 0, \quad \gamma \in (0,1),\ m > n.$$

The proofs are based on the classical Gaussian hypergeometric functions:

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}, \quad |z| < 1.$$

$$\frac{d^2 H_{n,n}}{d\gamma^2}(\gamma) = 2r(2r-1)\mathbf{E}\left[U^{2n}V^{2n}\left(\gamma(U^2+V^2)-V^2\right)^{2r-2}(U^2+V^2)^2\right] > 0,$$

$$\frac{dH_{n,n}}{d\gamma}\left(\frac{1}{2}\right) = 2r\mathbf{E}\left[U^{2n}V^{2n}\left(\frac{U^2-V^2}{2}\right)^{2r-1}(U^2+V^2)\right] = 0.$$

Then, $H_{n,n}(\gamma)$ reaches its unique minimum at $\gamma = \frac{1}{2}$. Hence it is sufficient to verify that $H_{n,n}\left(\frac{1}{2}\right) = 0$, i.e.,

$$\sum_{i=0}^{2r}(-1)^i\binom{2r}{i}\left(\frac{1}{2}\right)_{n+2r-i}\left(\frac{1}{2}\right)_{n+i} = 2^{2r}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_{n+r}.$$

Lemma Let $l, r \in \mathbb{N}$ satisfying $l \leq r$. Then we have

$$\sum_{i=0}^{l-1} \frac{\binom{2r}{i}\binom{l-1}{i}}{\binom{2r-l}{i}} = \frac{(2r)!}{2r!r!\binom{2r-l}{r}}.$$

The symmetric case: $H_{n,n}(\gamma) \geq 0$

$$
\begin{aligned}
& \sum_{i=0}^{l-1} \frac{\binom{2r}{i}\binom{l-1}{i}}{\binom{2r-l}{i}} \\
= \; & \sum_{i=0}^{l-1} \frac{(-2r)_i(1-l)_i}{(l-2r)_i \cdot i!}(-1)^i \\
= \; & \sum_{i=0}^{\infty} \frac{(-2r)_i(1-l)_i}{(l-2r)_i \cdot i!}(-1)^i \\
= \; & \lim_{\varepsilon \to 0} \sum_{i=0}^{\infty} \frac{(-2(r+\varepsilon))_i(1-l)_i}{(l-2(r+\varepsilon))_i \cdot i!}(-1)^i \\
= \; & \lim_{\varepsilon \to 0} \lim_{z \to -1} \sum_{i=0}^{\infty} \frac{(-2(r+\varepsilon))_i(1-l)_i}{(l-2(r+\varepsilon))_i \cdot i!} z^i
\end{aligned}
$$

The symmetric case: $H_{n,n}(\gamma) \geq 0$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\lim_{z\to -1} F(-2(r+\varepsilon), 1-l, (l-2(r+\varepsilon)); z) \\
&= \lim_{\varepsilon\to 0} \frac{\Gamma(l-2(r+\varepsilon))\Gamma(1-(r+\varepsilon))}{\Gamma(1-2(r+\varepsilon))\Gamma(l-(r+\varepsilon))} \\
&= \lim_{\varepsilon\to 0} \frac{(-(r+\varepsilon))\cdots(1-2(r+\varepsilon))}{(l-(r+\varepsilon)-1)\cdots(l-2(r+\varepsilon))} \\
&= \frac{(2r-1)\cdots r}{(2r-l)\cdots(r+1-l)} \\
&= \frac{(2r-1)!}{(r-1)!r!\binom{2r-l}{r}} \\
&= \frac{(2r)!}{2r!r!\binom{2r-l}{r}}.
\end{aligned}
$$

Lemma Let $l, r \in \mathbb{N}$ satisfying $l \leq r$. Then we have

$$\frac{2r!(l-1)!(2r-2l+1)!}{(2r)!(r-l)!} \sum_{i=0}^{l-1} \binom{2r}{i} \binom{2r-l-i}{2r-2l+1} = 1.$$

Corollary Let $l, r \in \mathbb{N}$ satisfying $l \leq r$. Then we have

$$\sum_{i=0}^{l-1} \frac{\binom{l-1}{i}}{\binom{2r-i}{l}} = \frac{1}{2\binom{r}{l}}.$$

Proof of identity:

$$\sum_{i=0}^{2r}(-1)^i\binom{2r}{i}\left(\frac{1}{2}\right)_{n+2r-i}\left(\frac{1}{2}\right)_{n+i} = 2^{2r}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_{n+r}.$$

Equivalent version:

$$\left\{\frac{2r!}{(2r)!}\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i}\left(\frac{1}{2}+n+r\right)_{r-i}\left(\frac{1}{2}+n\right)_i\right\}+\frac{(-1)^r}{r!}\left(\frac{1}{2}+n\right)_r = 1.$$

Define an r-th degree polynomial L by

$$L(x) = \left\{ \frac{2r!}{(2r)!} \sum_{i=0}^{r-1} (-1)^i \binom{2r}{i} (x+1+r)_{r-i} (x+1)_i \right\} + \left\{ \frac{(-1)^r}{r!} (x+1)_r \right\} - 1.$$

$L(0) = 0$. From the previous lemma, we get

$$L(-l) = 0, \quad l \in \{1, 2, \ldots, r\}.$$

Hence the r-th degree polynomial L has at least $(r+1)$ roots, which implies that $L \equiv 0$.

$$L\left(n - \frac{1}{2}\right) = 0.$$

The asymmetric case: $H_{m,n}(\gamma) > 0$

$$G_{m,n}(\gamma) = \mathbf{E}\left[U^{2m}V^{2n}\left(\gamma(U^2+V^2)-V^2\right)^{2r}\right], \quad \gamma \in \mathbb{R}.$$

$G_{m,n}$ is a strictly convex function on $\mathbb{R}$ and hence reaches its minimum at some $\gamma_m \in (0,1)$ with $\frac{d}{d\gamma}G_{m,n}(\gamma_m) = 0$.

Lemma For $0 < \gamma < 1$,

$$G_{m,n}(\gamma) = 2^{m+n+2r}\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_{n+2r} F\left(-2r, -m-n-2r, \frac{1}{2}-n-2r; \gamma\right).$$

Pfaff transformation

$$F(-2r, \frac{1}{2}+m, \frac{1}{2}-n-2r; -z) = (1+z)^{2r}F(-2r, -m-n-2r, \frac{1}{2}-n-2r; \frac{z}{1+z}).$$

Gauss' contiguous relations of hypergeometric functions. Consider the six functions

$$F(a \pm 1, b, c; z), \quad F(a, b \pm 1, c; z), \quad F(a, b, c \pm 1; z),$$

which are called contiguous to $F(a, b, c; z)$.

$$c(1-z)F - cF(a-1) + (c-b)zF(c+1) = 0,$$

$$(b-a)F + aF(a+1) - bF(b+1) = 0,$$

$$[c - 2b + (b-a)z]F + b(1-z)F(b+1) - (c-b)F(b-1) = 0,$$

$$\frac{d}{dz}F(a, b, c; z) = \frac{ab}{c}F(a+1, b+1, c+1; z).$$

For $0 < \gamma < 1$, define

$$B_m(\gamma) = F\left(-2r, -m-n-2r, \frac{1}{2}-n-2r; \gamma\right).$$

B_{m+1} reaches its minimum at some $\gamma_{m+1} \in (0,1)$ with $\frac{d}{d\gamma}B_{m+1}(\gamma_{m+1}) = 0$.

Lemma Let $m, n \in \mathbb{N} \cup \{0\}$, $r \in \mathbb{N}$ and $\gamma_{m+1} \in (0,1)$ be the minimum point of B_{m+1}. Then

$$B_{m+1}(\gamma_{m+1}) = B_m(\gamma_{m+1}).$$

Proof of $H_{m,n}(\gamma) > 0$ for $m > n$:

$H_{m,n}(\gamma) > 0$ can be written as

$$G_{m,n}(\gamma) > 2^{m+n+2r}\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_{n+r}\left(\frac{1}{2}\right)_r.$$

On the other hand,

$$G_{m,n}(\gamma) = 2^{m+n+2r}\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_{n+2r}B_m(\gamma).$$

Hence, we need show that

$$B_m(\gamma) > \left(\frac{1}{2}\right)_{n+r}\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_{n+2r}^{-1} = \binom{n+2r}{r}_{\frac{1}{2}}^{-1}.$$

In the symmetric case we have proved that

$$B_n(\gamma) > \binom{n+2r}{r}_{\frac{1}{2}}^{-1}, \quad \gamma \neq \frac{1}{2}.$$

By the previous lemma, we get

$$B_{n+1}(\gamma) \geq B_{n+1}(\gamma_{n+1}) = B_n(\gamma_{n+1}) > \binom{n+2r}{r}_{\frac{1}{2}}^{-1}.$$

Suppose the inequality holds for $m = k \geq n+1$. Using the previous lemma again, we get

$$B_{k+1}(\gamma) \geq B_{k+1}(\gamma_{k+1}) = B_k(\gamma_{k+1}) > \binom{n+2r}{r}_{\frac{1}{2}}^{-1},$$

Therefore, the proof is complete by induction.

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Lemma Suppose that (X, Y, Z) is a centered Gaussian random vector such that $\alpha X + \beta Y + \gamma Z = 0$ for some constants α, β, γ that are not all zero. Then for any $m, n \in \mathbb{N}$,

$$\mathbf{E}\left[X^{2m} Y^{2m} Z^{2n}\right] > \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2m}]\mathbf{E}[Z^{2n}].$$

The proof is based on the improved Gaussian product inequality given above.

We assume without loss of generality that $Z = X - Y$ and $\mathbf{E}[Z^2] = 1$.

Define

$$a = \mathbf{E}[XZ], \quad b = \mathbf{E}[YZ],$$

and

$$U = X - aZ = Y - bZ, \quad V = \sqrt{|ab|}Z.$$

To prove the desired inequality, it is sufficient to verify that for $0 \le i \le m$,

$$\mathbf{E}\left[V^{2n+2i}U^{2i}(V^2 - U^2)^{2m-2i}\right] > \binom{m}{i}_{\frac{1}{2}} \mathbf{E}[V^{2n+2i}]\mathbf{E}[U^{2i}]\left\{\mathbf{E}[(V+U)^{2m-2i}]\right\}^2.$$

Theorem Let (X, Y, Z) be a 3-dimensional centered Gaussian random vector. Then for any $m, n \in \mathbb{N}$,

$$\mathbf{E}\left[X^{2m}Y^{2m}Z^{2n}\right] \geq \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2m}]\mathbf{E}[Z^{2n}].$$

Proof. Define

$$Z_0 = \mathbf{E}\left[Z|X,Y\right], \quad Z_1 = Z - Z_0.$$

Then,

$$Z^{2n} = (Z_0 + Z_1)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} Z_0^{2n-i} Z_1^i.$$

Note that Z_1 is independent of X, Y. Hence

$$\mathbf{E}\left[Z_0^{2n-i} Z_1^i | X, Y\right] = Z_0^{2n-i}\mathbf{E}\left[Z_1^i\right],$$

which is equal to zero for odd i.

$$\mathbf{E}\left[Z^{2n}|X,Y\right] = \sum_{i=0}^{n} \binom{2n}{2i} Z_0^{2n-2i}\mathbf{E}\left[Z_1^{2i}\right].$$

Note that $Z_0 = \alpha X + \beta Y$ holds for some $\alpha, \beta \in \mathbb{R}$. Then, it follow from the previous lemma that

$$\mathbf{E}\left[X^{2m}\, Y^{2m}\, Z_0^{2n-2i}\right] \geq \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2m}]\mathbf{E}[Z_0^{2n-2i}].$$

Thus,

$$\begin{aligned}
\mathbf{E}\left[X^{2m}\, Y^{2m}\, Z^{2n}\right] &= \mathbf{E}\left[\mathbf{E}\left[Z^{2n}|X, Y\right] \cdot X^{2m} Y^{2m}\right] \\
&= \sum_{i=0}^{n} \binom{2n}{2i} \mathbf{E}\left[X^{2m}\, Y^{2m} Z_0^{2n-2i}\right] \mathbf{E}[Z_1^{2i}] \\
&\geq \sum_{i=0}^{n} \binom{2n}{2i} \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2m}]\mathbf{E}[Z_0^{2n-2i}]\mathbf{E}[Z_1^{2i}] \\
&= \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2m}] \sum_{i=0}^{n} \binom{2n}{2i} \mathbf{E}\left[Z_0^{2n-2i} Z_1^{2i}\right] \\
&= \mathbf{E}[X^{2m}]\mathbf{E}[Y^{2m}]\mathbf{E}\left[(Z_0 + Z_1)^{2n}\right].
\end{aligned}$$